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The Polynomial Identity Degree of Quantum Algebras at Roots of Unity

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Abstract

This review develops, in a self-contained analytical form, the theory of the polynomial identity (PI) degree of quantum algebras specialised at roots of unity, focusing on three model families: quantum affine space, quantum matrix algebras arising from the Faddeev–Reshetikhin–Takhtajan (FRT) construction, and reflection equation algebras. Building on the classical framework of Artin, Procesi and Posner for prime PI rings and on the roots-of-unity structure theory initiated by De Concini, Kac and Procesi, the review synthesises the algebraic, representation-theoretic and geometric characterisations of the PI degree into a single organising picture, and traces how each characterisation is computed in practice through iterated Ore extensions, PBW-type normal forms and the Azumaya locus of the centre. Comparative tables collect the known PI-degree formulas and the structure of the relevant centres; explicit worked examples and a set of mathematical exercises illustrate the rank-theoretic computation in the smallest nontrivial cases. The review closes with a discussion of open problems - higher-rank reflection equation algebras, non-type-A generalisations, and the classification of indecomposable (non-simple) finite-dimensional modules - that remain active areas of current research. Keywords: Polynomial identity degree; Quantum algebras; Roots of unity; Azumaya locus; Quantum affine space; Reflection equation algebras

1. Introduction

A ring R satisfies a polynomial identity if some nonzero noncommutative polynomial vanishes under every substitution of elements of R. Commutative rings satisfy the simplest such identity, the commutator

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$[X, Y] = 0$, and PI rings form the natural noncommutative generalisation retaining much of the finiteness theory of commutative algebra [27, 43, 44].

The modern theory dates to Kaplansky, Amitsur, Levitzki, Posner and Procesi in the 1950s–1970s [1, 27, 42, 43]. Its two anchoring results are:

- the **Kaplansky–Amitsur classification** of primitive PI rings as full matrix rings over division rings [27, 1], and
- the **Artin–Procesi characterisation** of Azumaya algebras [4].

The result most directly responsible for the PI phenomenon at roots of unity is a theorem of Artin and Procesi, building on an earlier theorem of Posner [42]: a prime ring that is finitely generated as a module of rank n over a central subring automatically satisfies the standard identity of degree $2n$, and is therefore PI [4].

For a prime PI ring R with centre Z and $F = \mathrm{Frac}(Z)$, the total ring of fractions $R \otimes_Z F$ is a finite-dimensional central simple F-algebra, hence isomorphic to $M_d(D)$ for an integer d and a central division F-algebra D by the Artin–Wedderburn and Brauer theories. The integer d is the **PI degree**, written $\text{PI-deg}(R)$ [43, 44].

Four equivalent descriptions of d are used interchangeably in the literature, and each supplies a different practical route to computing it. These descriptions—algebraic, identity-theoretic, representation-theoretic and rank-theoretic—will be used repeatedly in the subsequent sections when analysing quantum algebras at roots of unity.

2. Polynomial Identity Rings: Foundational Theory

A ring R satisfies a polynomial identity if some nonzero noncommutative polynomial vanishes under every substitution of elements of R. Commutative rings satisfy the simplest such identity, the commutator $[X, Y] = 0$, and PI rings form the natural noncommutative generalisation retaining much of the finiteness theory of commutative algebra [27, 43, 44].

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$$S_m(X_1, \dots, X_m) = \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) X_{\sigma(1)} \cdots X_{\sigma(m)}$$

vanishes identically on any such ring when $m = 2n$.

For a prime PI ring R with centre Z and $F = \operatorname{Frac}(Z)$, the total ring of fractions $R \otimes_Z F$ is a finite-dimensional central simple F-algebra, hence isomorphic to $M_d(D)$ for an integer d and a central division F-algebra D by the Artin–Wedderburn and Brauer theories; the integer d is the PI degree, written $\operatorname{PI-deg}(R)$ [43, 44]. Four equivalent descriptions of d are used interchangeably in the literature, and each supplies a different practical route to computing it.

Table 1. Four equivalent characterisations of the PI degree of a prime affine PI algebra R (K algebraically closed, Z the centre, $F = \operatorname{Frac}(Z)$).

Characterisation	Definition	Typical use
Algebraic	d with $R \otimes_Z F \cong M_d(D)$, D a central division F-algebra	Structural / Brauer-theoretic statements
Identity-theoretic	least d with S_{2d} an identity for R, S_{2d-2} not	Lower bounds via explicit non-identities
Representation-theoretic	maximal K-dimension of a simple R-module	Direct verification by exhibiting a module
Rank-theoretic	$\sqrt{n}$ where R is Z-free of rank n (Azumaya locus nonempty)	Combinatorial computation via PBW bases

These four viewpoints will be invoked throughout the review: the algebraic description appears when discussing central simple algebras obtained after localisation; the identity-theoretic description is useful for lower bounds; the representation-theoretic description is the most direct computational tool once simple modules are classified; and the rank-theoretic description is the one that emerges most naturally from the PBW bases of quantum algebras at roots of unity.

3. The De Concini–Kac–Procesi Framework

The systematic study of quantum algebras at roots of unity was initiated by De Concini and Kac [21], and extended jointly with Procesi [22], for the quantised enveloping algebra $U_\varepsilon(\mathfrak{g})$. Their central result is that $U_\varepsilon(\mathfrak{g})$ contains a large central Hopf subalgebra, isomorphic via a quantum Frobenius map to a classical coordinate ring, over which $U_\varepsilon(\mathfrak{g})$ is a free module of rank equal to the order of the Weyl group times a power of the order N of ε [21, 22, 34].

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This De Concini–Kac–Procesi (DKP) framework, together with Lusztig's parallel development of the quantum Frobenius homomorphism and small quantum groups [34], supplies the conceptual template that has since been transplanted, algebra by algebra, to quantum affine space, quantum matrix algebras, reflection equation algebras, quantised Weyl algebras, quantum cluster algebras and stated skein algebras [17, 19, 31, 35].

Recent work continues to refine the structure of the De Concini–Kac forms themselves [33]. The mechanism generalises cleanly. Write A_q for a quantum algebra depending on scalars q_{ij}, and suppose each q_{ij} has finite multiplicative order ℓ_{ij} once q is specialised. Monomials of the form $x_i^{\ell_i}$, where ℓ_i is typically the least common multiple of the relevant ℓ_{ij}, become central, because the accumulated commutation scalar raised to the ℓ_i-th power equals 1. The subalgebra Z that these monomials generate is commutative, and A_ε is shown—usually by exhibiting an explicit PBW-type spanning set of "reduced" monomials with bounded exponents—to be a free Z-module of finite rank n. The PI degree is then the rank-theoretic invariant $\sqrt{n}$, valid wherever the Azumaya locus of Z (the locus where the fibre algebra is a full matrix algebra) is nonempty, which the Artin–Procesi theorem guarantees whenever A_ε is prime [4].

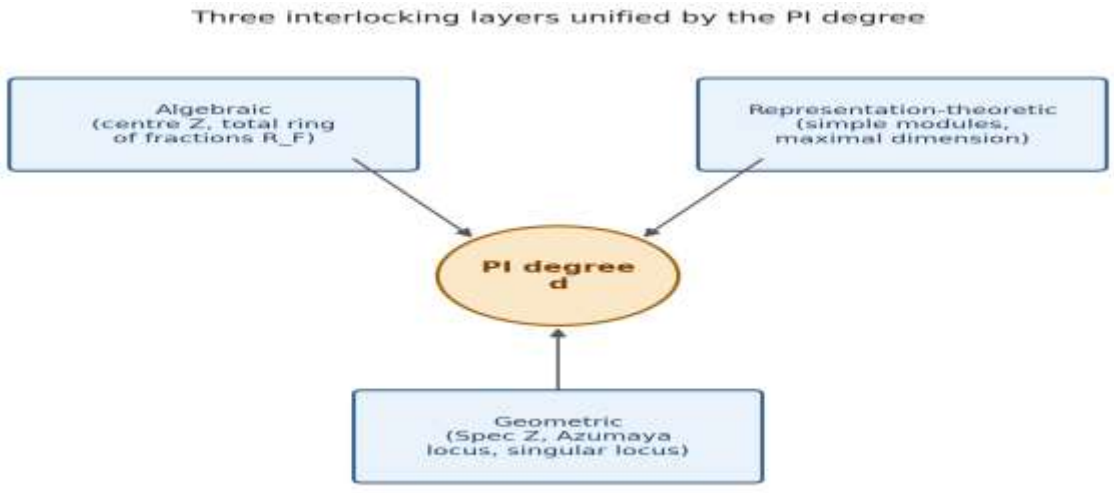


Figure 1. The PI degree as the invariant unifying the algebraic, representation-theoretic and geometric descriptions of a quantum algebra at a root of unity.

Three interlocking layers of structure recur throughout this pattern and are unified by the PI degree:

- an **algebraic layer**, concerning the primeness and centre of A_ε;

- a **representation-theoretic layer**, concerning the classification and dimensions of simple A_ε-modules;
- a **geometric layer**, in which Z is reinterpreted as the coordinate ring of an affine variety and the representation theory becomes a fibre-wise statement over that variety.

3.1 Iterated Ore Extensions as the Common Structural Framework

The rank computation that ultimately produces the PI degree is, in every case treated in this review, carried out through the theory of iterated Ore extensions. Given a K-algebra A, an automorphism σ of A and a σ-derivation δ (a K-linear map satisfying $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$), the Ore extension $A[x; \sigma, \delta]$ adjoins a new generator x subject to

$$xa = \sigma(a)x + \delta(a) \qquad \text{for } a \in A.$$

Iterating this construction n times, adjoining x_1, then x_2 to $K[x_1; \sigma_1, \delta_1]$, and so on, yields an algebra that is automatically a Noetherian domain over a field K and possesses a PBW-type basis of ordered monomials $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ [17, 25].

Quantum affine space is the case in which every σ_i is a diagonal scaling automorphism and every $\delta_i = 0$; quantum matrix algebras and reflection equation algebras require nontrivial derivation terms in at least some layers, reflecting the extra linear correction terms in their defining relations (the term $(q - q^{-1})bc$ in the $ad - da$ relation of Section 5 is the paradigm example) [23, 35].

Because all three families share this iterated-Ore-extension shape, a single body of technique—construction of the PBW basis, computation of Gelfand–Kirillov dimension, verification of primeness via general Ore-extension criteria, and, at roots of unity, identification of the new central elements and the resulting rank over the centre—applies uniformly, with family-specific modification, across the whole programme [17, 25].

Primeness of the specialised algebra A_ε is not automatic and must be checked at each stage; the standard sufficient criterion is that an iterated Ore extension of a prime ring by an automorphism of finite order, together with a compatible derivation, is prime under mild conditions on the automorphism and derivation, conditions that are routine, if occasionally lengthy, to verify for the specific automorphisms and derivations arising from the R-matrix relations underlying $\mathcal{O}_q(M_n)$ and L_q [17, 20]. Once primeness and PI-ness (via Artin–Procesi) are secured, the remaining work is purely combinatorial: exhibit a spanning set of normal-form monomials, prove linear independence over Z by a PBW-ordering argument, and read off the rank.

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4. Quantum Affine Space: A Complete Prototype

Quantum affine space $K_q[x_1, \dots, x_n]$ is generated by $x_1, \dots, x_n$ subject to

$$x_i x_j = q_{ij} x_j x_i \qquad \text{for } i < j,$$

where $q = (q_{ij})$ is multiplicatively skew-symmetric. It is the simplest possible iterated Ore extension—each generator is adjoined by a diagonal automorphism with no derivation term—and consequently the entire DKP mechanism can be carried out by hand [17, 35].

When each q_{ij} is specialised to a root of unity of order ℓ_{ij}, and ℓ_i denotes the least common multiple of the relevant ℓ_{ij} for fixed i, the powers $x_i^{\ell_i}$ are central, the subalgebra $Z = K[x_1^{\ell_1}, \dots, x_n^{\ell_n}]$ is a polynomial ring, and $K_q[x_1, \dots, x_n]$ is Z-free of rank $\ell_1 \ell_2 \cdots \ell_n$ on the reduced monomials $x_1^{a_1} \cdots x_n^{a_n}$ with $0 \le a_i < \ell_i$. The PI degree follows immediately:

$$\text{PI-deg}\big(K_q[x_1, \dots, x_n]\big) = \sqrt{\ell_1 \ell_2 \cdots \ell_n}.$$

This computation is classical and appears in many expositions [17, 38].

Worked example ($n = 2$). Take a single scalar $q_{12} = q$ of order N. Then $\ell_1 = \ell_2 = N$, $Z = K[x_1^N, x_2^N]$, the algebra is Z-free of rank N^2, and PI-deg $= N$. The maximal-dimensional simple modules are the N-dimensional "Weyl-type" modules spanned by a cyclic vector $v, x_2 v, x_2^2 v, \dots, x_2^{N-1} v$, with x_1 permuting this basis by a scalar shift—the direct prototype for every higher-rank computation that follows in this review [38].

Because $Z \cong K[y_1, \dots, y_n]$ with $\mathrm{Spec}(Z)$ an affine n-space, the Azumaya locus is exactly the algebraic torus $(\mathbb{K}^\times)^n \subset \mathbb{A}^n$ where every $y_i \neq 0$, and its complement—the coordinate hyperplanes $\{y_i = 0\}$—is the locus of sub-maximal simple modules.

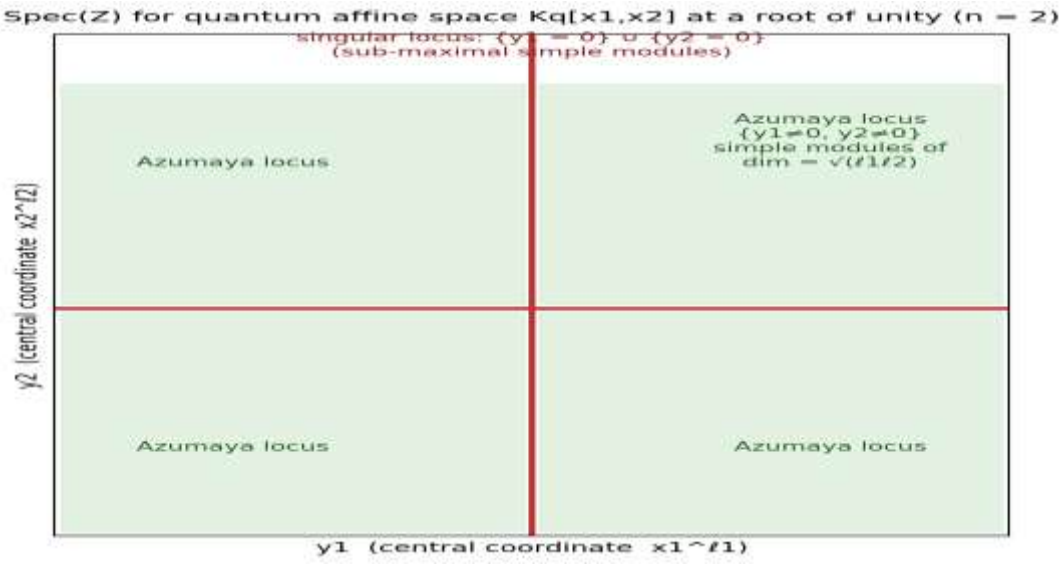


Figure 2. Spec(Z) for $K_q[x_1, x_2]$ at a root of unity of order N: the Azumaya locus is the open torus; the coordinate axes form the singular locus.

This picture is the cleanest illustration available anywhere in the theory of the general Azumaya-locus dichotomy discussed in Section 7 below, and it is for this reason that quantum affine space is invariably treated first, as a calibration case, before the considerably more delicate computations for quantum matrix algebras and reflection equation algebras are attempted. Related constructions of simple modules over two-parameter quantised algebras continue to appear [39].

5. Quantum Matrix Algebras and the FRT Construction

The quantised coordinate ring $\mathcal{O}_q(M_n(K))$ of $n \times n$ matrix space is generated by n^2 entries x_{ij} subject to the FRT relation

$$RX_1X_2 = X_2X_1R,$$

where $R \in \mathrm{End}(K^n \otimes K^n)$ is the R-matrix associated with $U_q(\mathfrak{sl}_n)$ and $X_1 = X \otimes I$, $X_2 = I \otimes X$ for $X = (x_{ij})$ [23, 19].

For $n = 2$, writing $a = x_{11}$, $b = x_{12}$, $c = x_{21}$, $d = x_{22}$, the relations reduce to the familiar six scalar identities Setting $q = 1$ recovers the ordinary commutative coordinate ring of 2×2 matrices, and $\mathcal{O}_q(GL_n)$, $\mathcal{O}_q(SL_n)$ are obtained by inverting, respectively setting to 1, the quantum determinant D_q [19, 31].

Structurally, $\mathcal{O}_q(M_n(K))$ is again presentable as an iterated Ore extension—now with genuine derivation terms in the mixed relations, reflecting the $(q - q^{-1})bc$ correction term—and this presentation is the technical route by which primeness, Gelfand–Kirillov dimension, and a PBW-type spanning set are established [17, 25].

At a primitive N-th root of unity, the quantum determinant and appropriate powers of the generators generate an enlarged centre Z, and the algebra becomes finite, indeed free, over Z. In the smallest case $n = 2$, the quantum determinant $D_q = ad - q\,bc$ is already central for generic q; at a primitive N-th root of unity, $\mathcal{O}_q(M_2(K))$ is Z-free of rank N^4, giving

$$\mathrm{PI\text{-}deg}\left(\mathcal{O}_q(M_2(K))\right) = N^2.$$

This computation, together with the classification of simple modules, has been revisited and extended in recent work [14, 15].

This $n = 2$ computation is the standard sanity check against which general-rank formulas are tested, and it already illustrates a recurring theme of the subject: the PI degree of the matrix-type algebra grows quadratically in N where the affine-space analogue grows linearly, reflecting the doubling of the number of independent "quasi-commutative" directions once a genuine R-matrix, rather than a diagonal scalar matrix, governs the relations.

For general n the rank computation is considerably more delicate, because the

enlarged centre involves not only powers of the individual generators but also quantum minors $\Delta_{I,J}$—q-deformed Laplace-expansion analogues of ordinary minors satisfying quantum Plücker relations—and identifying the Azumaya locus requires understanding where these quantum minors, and not merely the generators themselves, fail to vanish. Related results on quantum Grassmannians and Schubert varieties at roots of unity appear in [8].

6. Reflection Equation Algebras

Reflection equation algebras L_q arose from the study of quantum integrable systems with a reflecting boundary, and are generated by the entries of a matrix $L = (l_{ij})$ subject to the reflection equation

$$RL_1RL_1 = L_1RL_1R$$

in tensor-leg notation. They may be realised concretely as subalgebras of tensor powers of $\mathcal{O}_q(M_n)$, from which they inherit a comodule-algebra structure over the ambient quantised coordinate ring—a symmetry not manifestly present in the plain quantum matrix algebra, and one that has been exploited to transport semisimplicity and primeness criteria from $\mathcal{O}_q(G)$ to L_q [20].

Like $\mathcal{O}_q(M_n)$, L_q admits an iterated-Ore-extension presentation, making the same PBW-basis and rank-computation strategy available, suitably adapted to the reflection-equation relations.

At a root of unity, L_q acquires a large centre generated by powers of the l_{ij} together with quantum-trace and higher quantum-power-sum invariants—the reflection-equation analogues of the classical ring of GL_n-conjugation-invariant polynomials on matrix space—and becomes finite over this centre.

Cooney, Ganev and Jordan established foundational results on quantum Weyl algebras and reflection equation algebras at roots of unity [20], and subsequent work by Bera and Mukherjee has extended the PI-degree computation and the classification of simple modules to further cases, including the Dipper–Donkin quantised matrix algebra and its reflection-equation counterpart [13, 14, 15]. Further results on simple modules over related quantised Weyl algebras appear in [10, 11, 12].

Because the fibre algebras attached to non-Azumaya (singular) central characters of L_q frequently fail to be semisimple, reflection equation algebras are also a natural setting for constructing genuinely indecomposable (non-simple) finite-dimensional modules—extensions between distinct simple modules, or self-extensions—a topic that extends the classification problem beyond the semisimple level and remains comparatively less developed than the corresponding classification of simple modules.

It bears emphasis that type-A reflection equation algebras, associated with GL_n or SL_n, are only one member of a wider family: analogous constructions exist for

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the orthogonal and symplectic quantum groups (types B, C, D), governed by an additional "unitarity" condition on L relative to the relevant quantum bilinear form. The PI-degree theory for these non-type-A reflection equation algebras is considerably less complete, and constitutes one of the clearer open directions surveyed in Section 9. Related work on quantum symplectic spaces appears in [37].

7. Azumaya Loci and the Geometry of Representation Varieties

The centre Z of a quantum algebra A_ε at a root of unity is, in every case considered here, a finitely generated commutative K-algebra, so that $\mathrm{Max}(Z)$ is an affine variety functioning as a parameter space—a representation variety—for the finite-dimensional representation theory of A_ε. The Azumaya locus, by general PI theory, is a nonempty Zariski-open subset of the irreducible variety $\mathrm{Spec}(Z)$ whenever A_ε is prime PI [4]; its complement, the singular or ramification locus, is a proper closed subvariety whose dimension and singularities govern the complexity of the non-generic representation theory.

For quantum affine space this picture is fully explicit, as Section 4 showed: the singular locus is a union of smooth coordinate hyperplanes meeting transversally. For quantum matrix algebras and reflection equation algebras, Z is considerably more elaborate—generated not only by central powers of generators but by quantum minors, the quantum determinant, and quantum-trace invariants—and $\mathrm{Spec}(Z)$ correspondingly reflects the classical invariant theory of the GL_n-conjugation action on matrix space, of which the reflection-equation-algebra centre is, heuristically, a quantisation.

Mapping the Azumaya locus explicitly in these more elaborate cases—determining precisely which vanishing loci of quantum minors and quantum-trace invariants cut out the singular locus—is one of the more demanding tasks in the current literature, and is where consistency checks become valuable:

- the Krull dimension of Z should equal the Gelfand–Kirillov dimension of A_ε (since A_ε is Z-finite),
- the rank computed via the PBW spanning set should match the degree of the Azumaya locus as a covering of $\mathrm{Spec}(Z)$, and
- the singular locus should have strictly smaller dimension than $\mathrm{Spec}(Z)$ itself, reflecting that "most" central characters are Azumaya.

A closely related theme, treated systematically in recent work on skein algebras and quantum cluster algebras, is that the Azumaya locus can frequently be characterised as the non-vanishing locus of a single distinguished central element—a quantum-cluster discriminant, or a quantum-trace-type invariant—generalising the role played by the ordinary determinant in the classical statement that the singular locus of the space of matrices

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is exactly the vanishing locus of the determinant [26, 36, 41, 28].

8. Comparative Analysis

Table 2. PI-degree formulas for the three core algebra families (rank-2 / simplest nontrivial cases shown; N = order of the relevant root of unity).

Algebra	Centre Z (typical generators)	Z-rank of A_ε	PI degree
Quantum affine space $K_q[x_1, \dots, x_n]$	$x_1^{\ell_1}, \dots, x_n^{\ell_n}$	$\ell_1 \ell_2 \cdots \ell_n$	$\sqrt{\ell_1 \cdots \ell_n}$ $(= N$ for $n = 2, \ell_1 = \ell_2 = N)$
Quantum matrix algebra $\mathcal{O}_q(M_2(K))$	$a^N, b^N, c^N, d^N, D_q^N$ (quantum determinant)	N^4	N^2
Reflection equation algebra L_q (type A, rank 2)	powers of l_{ij}, quantum trace / power-sum invariants	case-dependent; established for low rank	N^2 in the low-rank cases studied to date

Table 3. Related quantum structures to which the roots-of-unity / PI-degree template has recently been applied.

Structure	Representative recent study	Key mechanism
Quantised enveloping algebra $U_\varepsilon(\mathfrak{g})$	De Concini, Kac & Procesi (foundational) [21, 22]; Losev, Tsymbaliuk & Vu (2026) [33]	Quantum Frobenius map into a central Hopf subalgebra
Quantum cluster algebras	Nguyen, Trampel & Yakimov (2025) [41]; Huang, Lê & Yakimov (2021) [26]	Non-commutative discriminant locates the Azumaya locus
Stated skein algebras	Karuo & Korinman (2026) [28]; Karuo & Wang (2026) [30]; Frohman, Kania-Bartoszynska & Lê (2025) [24]	Dense centre parametrising classical character varieties
Quantised Weyl / Heisenberg algebras	Bera & Mukherjee (2023) [13]; Bera, Mandal & Nandy (2024) [12]	PBW spanning set over a polynomial centre
Quantum symmetric pairs	Bao & Sale (2021) [5]; Song & Zhang (2026) [45]	Frobenius-twisted central subalgebra from the ambient $U_\varepsilon(\mathfrak{g})$
Cluster / Poisson orders more generally	Muller, Nguyen, Trampel & Yakimov (2024) [36]; Brown & Yakimov (2024) [18]	Poisson-geometric description of the Azumaya locus

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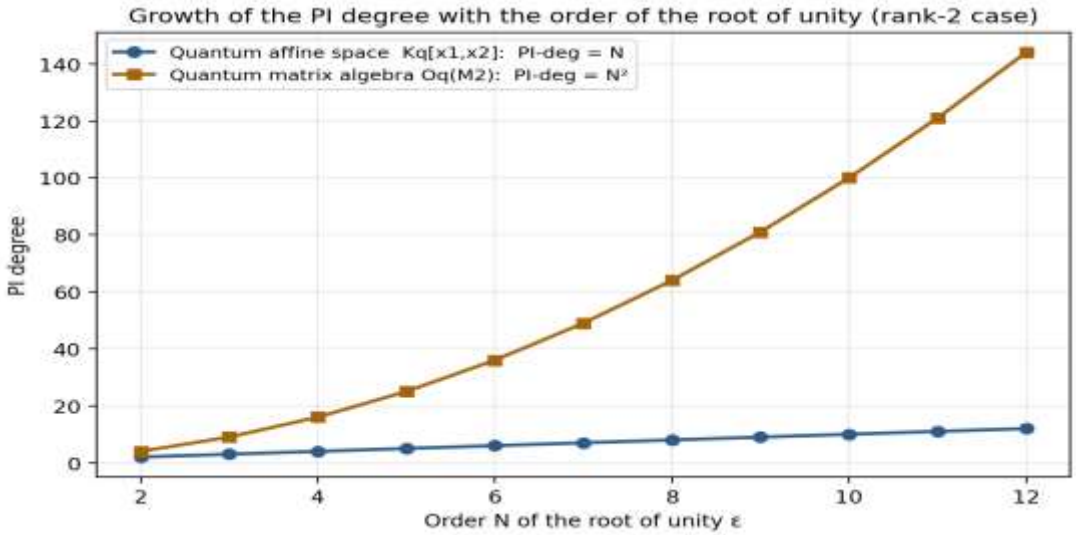


Figure 3. PI-degree growth with the order N of the root of unity for the rank-2 quantum affine space (linear) versus the rank-2 quantum matrix algebra (quadratic).

A consistent pattern emerges from this comparison. Algebras whose defining relations are purely "quasi-commutative" (quantum affine space, and more generally quantum tori and their variants) admit centres that are themselves polynomial or Laurent-polynomial rings, Azumaya loci that are literal algebraic tori, and PI-degree formulas that are simple products or square roots of the orders of the individual commutation scalars [17, 38].

Algebras whose relations genuinely mix generators through an R-matrix (quantum matrix algebras, reflection equation algebras, quantum symmetric pairs) have centres generated by higher-degree invariants—quantum minors, quantum determinants, quantum traces—whose vanishing loci are typically singular subvarieties rather than coordinate hyperplanes, and correspondingly the PI-degree formulas, where known, tend to be quadratic or higher-degree functions of the order N of the root of unity rather than linear ones, and their derivation requires substantially more combinatorial labour in constructing and verifying a PBW-type spanning set [14, 20, 5].

9. Open Problems and Directions for Future Work

Several problems remain open or only partially resolved, and are worth flagging explicitly as directions for continuation of the programme surveyed here.

1. The PI degree of the quantum matrix algebra $\mathcal{O}_q\big(M_n(K)\big)$ and the type-A reflection equation algebra for general rank n, and for arbitrary (not necessarily coprime) N, has not been given a fully closed-form treatment on a par with the $n = 2$ case; the combinatorics of the quantum-minor generators of the

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centre grows rapidly with n, and a uniform PBW-basis argument valid for all n remains a substantial undertaking [14, 15].

2. Reflection equation algebras of types B, C, D—associated with the orthogonal and symplectic quantum groups rather than GL_n—carry an additional unitarity condition on the generating matrix L, and the corresponding PI-degree theory has, at the time of writing, been developed only in outline [37].
3. The classification of finite-dimensional indecomposable (non-simple) A_ε-modules, as opposed to simple modules, is considerably less complete than the classification of simples: composition series, extension groups, and self-extensions at non-Azumaya central characters have been worked out only for selected low-rank examples, and a systematic theory—analogous to the Auslander–Reiten theory available for finite-dimensional algebras more generally—is largely absent from the quantum-algebra literature [20, 14].
4. The relationship between the PI degree and the Poisson geometry of the semiclassical limit—the Poisson order perspective developed by Brown, Goodearl, Yakimov and collaborators for cluster algebras and more general algebras with a compatible Poisson structure—has been established in several individual cases but not yet as a general theorem applicable uniformly to quantum affine space, quantum matrix algebras and reflection equation algebras simultaneously [18, 36].

Table 4. Open problems in the PI-degree theory of quantum algebras at roots of unity.

Open problem	State of the art	Principal obstacle
Closed-form PI degree of $\mathcal{O}_q(M_n(K))$ for general n	Established for $n = 2,3$ and special N	Combinatorial growth of quantum-minor generators of Z
PI degree of types B, C, D reflection equation algebras	Framework outlined; explicit computations largely absent	Extra unitarity relations complicate the Ore-extension presentation
Classification of indecomposable (non-simple) modules	Worked out for selected low-rank fibre algebras	No general Auslander–Reiten-type theory available
Uniform Poisson-order description of the Azumaya locus	Established case-by-case (cluster algebras, some Weyl algebras)	Lack of a single Poisson structure compatible with all three families

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10. Beyond the Three Core Families

The last several years have seen the rank-theoretic PI-degree method applied well beyond the three core families reviewed here, and a brief survey of this wider literature helps calibrate how general the underlying mechanism really is.

Bellamy and collaborators examined invariant subalgebras of rational Cherednik algebras, showing that the symplectic-leaf structure of the associated Poisson variety parametrises the maximal-dimensional simple modules, so that the geometry of the centre and the PI degree are again two descriptions of the same underlying stratification [9].

Nguyen, Trampel and Yakimov developed a discriminant-based approach to root-of-unity quantum cluster algebras, in which the non-commutative discriminant of the cluster algebra is shown to vanish exactly on the complement of the Azumaya locus, giving an effective algorithmic route to the PI degree that avoids an explicit PBW-basis computation in cases where the discriminant can be evaluated directly [41, 26].

Frohman, Kania-Bartoszynska and Lê, together with Karuo, Korinman, Moon and Wong in a series of related papers, transplanted essentially the same machinery to stated and sliced skein algebras of surfaces, where the centre at a root of unity is shown to be dense and to parametrise the classical SL_2- or SL_n-character variety of the surface, so that the PI degree of the skein algebra becomes a bridge between quantum topological invariants and classical representation varieties [24, 16, 28, 29, 30].

Closer to the specific families treated in this review, Bera and Mukherjee have produced a sustained programme of PI-degree computations for quantised Weyl algebras, quantum Heisenberg enveloping algebras, and—most directly relevant here—the Dipper–Donkin variant of the quantised matrix and reflection equation algebras, establishing simple-module classifications and Azumaya-locus descriptions case by case [10, 11, 12, 13, 14, 15].

Launois, Lopes and Rogers introduced a deleting-derivations algorithm, adapted from the theory of quantum nilpotent algebras, that streamlines the identification of the centre at a root of unity for algebras admitting a suitable Ore-extension presentation, offering a partially algorithmic alternative to the by-hand PBW arguments used in the classical treatments [32, 8]. Related work on quantum graph algebras appears in [6, 7].

Weight representations of affine Kac–Moody algebras and small quantum groups continue to be studied in depth [3], as do path descriptions for characters of restricted quantum loop algebras [2] and representations of quantum symmetric pairs at roots of unity [5, 45]. Quiver varieties remain a powerful geometric tool for finite-dimensional representations of quantum affine algebras [40].

Taken together, this body of work indicates that the three-layer picture of Figure 1—algebraic finiteness, representation-theoretic maximality, geometric stratification—is not an idiosyncrasy of quantum affine space, quantum matrix algebras and reflection equation algebras, but a structural feature shared by essentially every quantum algebra admitting an iterated Ore-extension presentation and a compatible roots-of-unity

specialisation, which is precisely why the present review treats the three core families as representative case studies within a considerably larger landscape [17, 19, 31].

11. Conclusion

The PI degree functions, across the whole family of quantum algebras surveyed in this review, as the single invariant that ties together the algebraic finiteness of a quantum algebra over its centre at a root of unity, the maximal dimension attained by its finite-dimensional simple modules, and the geometry of the Azumaya locus inside the spectrum of that centre. For quantum affine space the computation is essentially complete and admits a fully explicit geometric picture, in which the PI degree is a simple product of the orders of the individual commutation scalars and the Azumaya locus is literally an algebraic torus. For quantum matrix algebras and reflection equation algebras, the same template applies in outline, but the presence of quantum minors, quantum determinants and quantum-trace invariants among the generators of the centre makes both the rank computation and the identification of the Azumaya locus considerably more delicate, and a fully uniform treatment for arbitrary rank remains an active area of ongoing research, as does the classification of finite-dimensional indecomposable modules beyond the semisimple level. The comparative tables and worked exercises assembled here are intended to make this body of theory, and the rank-theoretic method that underlies almost all of its computations, accessible as a single coherent analytical account rather than as a scattered collection of individually specialised results.

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